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LETTER TO THE EDITOR

Statistics of interior current distributions in two-dimensional open chaotic billiards

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Abstract
The probability current statistics of two-dimensional open chaotic ballistic billiards is studied both analytically and numerically. Assuming that the real and imaginary parts of the scattering wave function are both random Gaussian fields, we find a universal distribution function for the interior probability current. As a by-product we recover previous analytic forms for wave function statistics. The expressions bridge the entire region from GOE to GUE type statistics. Our analytic expressions are verified numerically by explicit quantum mechanical calculations of transport through a Bunimovich billiard.

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1. Introduction

For a quantum chaotic closed system it is well known that the statistical properties of the energy levels are described by random matrix theory (RMT) [1]. They follow the Gaussian orthogonal ensemble (GOE) and the Gaussian unitary ensemble (GUE), depending on whether time-reversal symmetry (TRS) of a system is preserved or not. In the same way the wave function statistics obeys different laws in the two cases. Let the scaled local density be $\rho(\boldsymbol{r}) = A|\psi(\boldsymbol{r})|^2$ where $\psi(\boldsymbol{r})$ is the normalized wave function and A is the area (volume) of the system. As prescribed by GOE the probability distribution is the well known Porter–Thomas (PT) distribution $P(\rho) = \left(1/\sqrt{2\pi\rho}\right)\exp(-\rho/2)$ when TRS is present (the Hamiltonian H is invariant under time reversal $\hat{T}(t \to -t)$ and ψ may be chosen real). On the other hand, the distribution takes the exponential Rayleigh form $P(\rho) = \exp(-\rho)$ as described by GUE when TRS is broken (H is not invariant with respect to $\hat{T}$ and ψ must be complex). It is easy to understand qualitatively why the statistics are so different in the two cases; for example, why small values of ρ have a much larger weight in GOE than in GUE. In the first case the real wave function vanishes along nodal lines in two-dimensional (2D) systems (surfaces in 3D). On the other hand, ψ is complex in the second case and vanishes only at nodal points (lines in 3D) resulting in less probability for small ρ. Depending on the relative weights of the orthogonal real and imaginary parts of ψ one can also define intermediate statistics that applies to the entire crossover region from GOE to GUE [2, 3].

We now consider what happens when the system is made open, for example, by attaching electron leads to some exterior reservoirs and a stationary current through the system is induced by applying suitable voltages to the reservoirs. The additional flexibility gained in this way leads to a number of interesting cases for the wave function statistics. Let us first look at the case when there is no current flow. The statistics will then be the same as for the closed system above; i.e., the kind of statistics simply depends on whether the Hamiltonian is invariant under $\hat{T}$ or not. On the other hand, if there is a stationary current via the leads we have to deal with a scattering wave function. This function, which must be complex, is written in 2D as

$$\psi(x, y) = u(x, y) + \mathrm{i}v(x, y). \tag{1}$$

and satisfies $(\nabla^2 + k^2)\psi = 0$. Even if the Hamiltonian itself is invariant under $\hat{T}$, the statistics will not follow GOE since the scattering wave function is complex. Because of the boundary conditions associated with the scattering wave function, it is no longer an eigenstate of the usual time-reversal operator $\hat{C}\hat{T}$ where $\hat{C}$ is complex conjugation of $\psi(\boldsymbol{r})$. We will show how the two Gaussian random fields u and v are identified to recover the intermediate statistics discussed above and how a universal distribution for the probability current density can be found. We will also compare theory with explicit numerical calculations for an open Bunimovich stadium.

2. Theory

In the following derivation of the wave function and probability current statistics we assume that the real and imaginary parts of ψ can be viewed as two independent isotropic Gaussian fields. An explicit example of such a state is given in [4] in the form of a Berry-type wave-chaotic function. In general, the assumption of independent fields can only make sense if we first extract a common phase factor. This feature will turn out to be most useful. Let us introduce the notation

$$\langle u^2 \rangle = \sigma_u^2 \qquad \langle v^2 \rangle = \sigma_v^2 \qquad \langle uv \rangle = \gamma \tag{2}$$

$$\sigma^2 = \sigma_u^2 + \sigma_v^2 = \langle |\psi|^2 \rangle \qquad \langle u \rangle = 0 \qquad \langle v \rangle = 0. \tag{3}$$

We define the averages as

$$\langle ... \rangle = \frac{1}{A} \int_A \mathrm{d}^2\boldsymbol{r} \; ... \tag{4}$$

where A is the area to be sampled. In our case it will be area of the cavity, but in principle it could be any area that one may wish to specify. In what follows we assume that wave function $\psi(\boldsymbol{r})$ is normalized:

$$\int_A \mathrm{d}^2\boldsymbol{r} \, |\psi(\boldsymbol{r})|^2 = 1 \tag{5}$$

and therefore $\sigma^2 A = 1$. To bring ψ to a 'diagonal' form in which the real and imaginary parts are independent Gaussian fields we introduce the new functions $p(x, y)$ and $q(x, y)$ by changing the phase as

$$\psi(x, y) \to \mathrm{e}^{-\mathrm{i}\alpha}\psi(x, y) = p(x, y) + \mathrm{i}q(x, y). \tag{6}$$

The condition $\langle pq\rangle = 0$ now allows us to determine α. By this step we are also able to find analytic expressions for the wave function and probability current statistics. Straightforward algebra gives

$$\begin{aligned}\tan 2\alpha &= \frac{2\gamma}{\sigma_u^2 - \sigma_v^2}\\ \langle p^2\rangle &= \tfrac{1}{2}\left[\sigma^2 + \sqrt{\sigma^4 - 4(\sigma_u^2\sigma_v^2 - \gamma^2)}\right]\\ \langle q^2\rangle &= \tfrac{1}{2}\left[\sigma^2 - \sqrt{\sigma^4 - 4(\sigma_u^2\sigma_v^2 - \gamma^2)}\right].\end{aligned} \tag{7}$$

Next let us consider the cumulative distribution $G(\rho)$ for the scaled density $\rho(\boldsymbol{r}) = A|\psi(\boldsymbol{r})|^2$:

$$G(\rho) = \int_{C(\rho)} f(p,q)\,\mathrm{d}p\,\mathrm{d}q. \tag{8}$$

The integration is defined by the circle $C(\rho)$ in the (p,q)-plane centred at the origin and with radius $\sqrt{\rho(\boldsymbol{r})}$; i.e., $(p^2+q^2)/\sigma^2 \leqslant \rho(\boldsymbol{r})$ in the integral above. The function $f(p,q)$ is the joint distribution for the random Gaussian fields p and q:

$$f(p,q) = \frac{1}{2\pi\sqrt{\langle p^2\rangle\langle q^2\rangle}}\exp\left[-\frac{1}{2}\left(\frac{p^2}{\langle p^2\rangle} + \frac{q^2}{\langle q^2\rangle}\right)\right]. \tag{9}$$

After integration of equation (8) we obtain

$$G(\rho) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-\exp[-\rho\mu(\mu+\nu\cos\theta)]}{\mu+\nu\cos\theta}\,\mathrm{d}\theta \tag{10}$$

where we have introduced the following notation:

$$\mu = \frac{1}{2}\left(\frac{1}{\epsilon}+\epsilon\right) \qquad \nu = \frac{1}{2}\left(\frac{1}{\epsilon}-\epsilon\right) \qquad \epsilon = \sqrt{\frac{\langle q^2\rangle}{\langle p^2\rangle}}. \tag{11}$$

Differentiating (10) with respect to ρ we find the final expression for the density distribution

$$P(\rho,\epsilon) = \mu\exp(-\mu^2\rho)I_0(\mu\nu\rho) \tag{12}$$

where $I_0(z)$ is the modified Bessel function of zeroth order.

The distribution in equation (12) coincides with the results obtained from RMT for closed systems [2, 3] and is therefore not new. For weakly open systems with point contacts Šeba *et al* [5] have related the statistical properties of the scattering matrix elements with the distribution $P(\rho)$ and have obtained the expression above. A derivation analogous to ours is also found in [4]. Our way of deriving equation (12), however, explicitly shows how to identify the two independent random fields in a given wave function. For the random wave function in equation (6) the limits $\langle q^2\rangle \to 0$, $\epsilon \to 0$ correspond to a time-reversed symmetric closed system and as a consequence one recovers the PT distribution and GOE statistics. On the other hand, the case $\langle p^2\rangle \to \langle q^2\rangle$, $\epsilon \to 1$ corresponds in this context to an open system through which there is a current flow. Consequently one finds the exponential Rayleigh distribution that corresponds to the GUE statistics. We have recently verified this type of crossover in wave function statistics for a Bunimovich stadium using numerical scattering methods [6]. In the crossover region the value of ϵ is obtained numerically using equations (2), (3), (7) and (11); i.e., ϵ is not merely a fitting parameter. This procedure also removes any ambiguities in α that might be present in computed or measured data.

In view of all of the previous work on the generic form of wave function statistics in chaotic systems it is surprising that no attention has been paid to the corresponding current distributions, except [4] which relates the average of the squared current to $\langle\rho\rangle$. Since currents may be measured [7–9] it is of interest to establish a form also for currents. Below we will show how to find a useful form that is both simple and universal. Let us limit ourselves to the case of a weak *net* current between narrow input and output leads. Inside the cavity, however, there will be a rich, whirling flow pattern, which is strongly influenced by the vortical motions around the nodal points associated with the complex form of the wave function. Hence the net current through the billiard turns out to be only a tiny fraction of the total internal flow, and particularly so for asymmetric arrangements of leads and wavelengths that are small compared to the dimensions of the cavity. As a result, the corresponding distributions may to a good approximation be chosen to be isotropic. Hence the components of the current effectively average to zero. These assumptions are verified by the numerical calculations to be discussed in the next section.

Our complex wave function (1) carries the probability current density ($\hbar = m = 1$)

$$\boldsymbol{j} = \mathrm{Im}(\psi^*\nabla\psi) = p\nabla q - q\nabla p. \tag{13}$$

To find the corresponding distribution it is convenient to begin with a characteristic function for the components of the probability current density

$$\Theta(\boldsymbol{a}) = \langle \exp[\mathrm{i}\boldsymbol{a}\cdot\boldsymbol{j}]\rangle = \langle \exp[\mathrm{i}(p\boldsymbol{a}\cdot\nabla q - q\boldsymbol{a}\cdot\nabla p)]\rangle. \tag{14}$$

Since $\langle p\nabla q\rangle = \langle q\nabla p\rangle = 0$ for isotropic fields ∇p and ∇q are statistically independent of p and q. They have the same distribution as in equation (9) with dispersions $\langle(\nabla p)^2\rangle = k^2\langle p^2\rangle$ and $\langle(\nabla q)^2\rangle = k^2\langle q^2\rangle$ which follows from the Schrödinger equation. Using the relation $\langle(\boldsymbol{a}\nabla p)^2\rangle = a^2k^2\langle p^2\rangle/2$ and similarly for ∇q, we obtain

$$\Theta(a) = \frac{1}{1+\tau^2 a^2} \tag{15}$$

where $a = |\boldsymbol{a}|$ and

$$\tau^2 = k^2\langle p^2\rangle\langle q^2\rangle/2. \tag{16}$$

From equation (15) it is now easy to calculate the distribution functions. For the components we have

$$\begin{aligned} P(j_x) &= \left\langle \delta\left(j_x - p\frac{\partial q}{\partial x} + q\frac{\partial p}{\partial x}\right)\right\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta(|a_x|)\,\exp(-\mathrm{i}a_x j_x)\,\mathrm{d}a_x \\ &= \frac{1}{2\tau}\exp(-|j_x|/\tau) \end{aligned}$$

and the same for $P(j_y)$. In order to derive the distribution function for the absolute value of the probability current density let us consider the joint distribution function

$$\begin{aligned} P(j_x, j_y) &= \frac{1}{2\pi}\int_0^{\infty} a\,J_0(aj)\Theta(a)\,\mathrm{d}a \\ &= \frac{1}{2\pi\tau^2}K_0\left(\frac{j}{\tau}\right) \end{aligned} \tag{17}$$

where $j = |\boldsymbol{j}|$ and $K_0(z)$ is the modified Bessel function of the second kind. Since this expression is radially symmetric one can find the probability density function $P(j)$ for j by just multiplying equation (17) with a factor of $2\pi j$. This gives us the final expression

$$P(j) = \frac{j}{\tau^2}K_0\left(\frac{j}{\tau}\right). \tag{18}$$

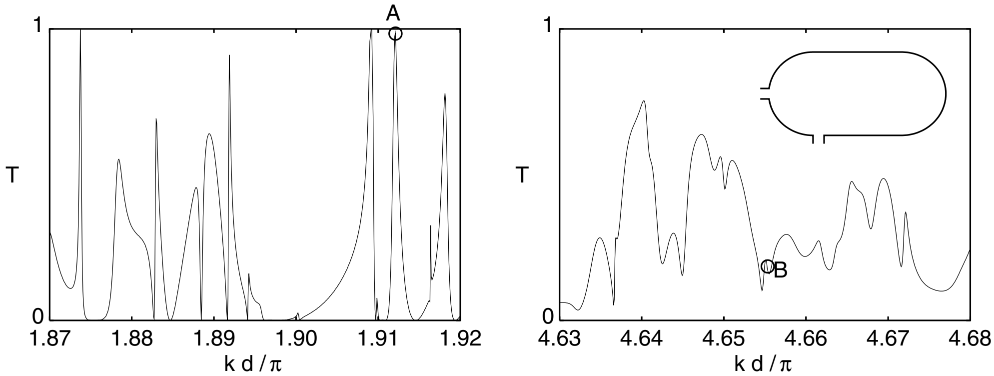


Figure 1. Transmission probability T as a function of Fermi wave number k for the open stadium billiard: (A) a low energy case with one open channel in the leads ($n = 1$), (B) a high energy case with $n = 4$. The inset shows the hard-wall Bunimovich stadium and the positions of the leads.

3. Numerical results

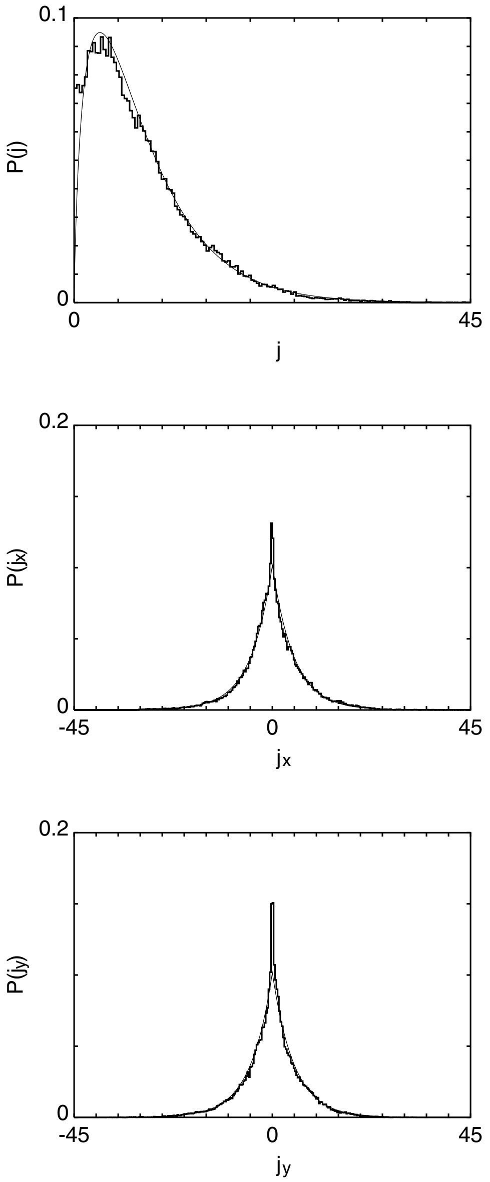
As a numerical verification of the analytic expressions for the probability current distribution, we consider an open 2D Bunimovich hard-wall stadium (see inset in figure 1). It is characterized by the radius of a semicircle a and the half-length of a straight section l, and coupled to a pair of leads with a common width d. Here we choose ($a = l$) and ($d/\sqrt{A} = 0.0935$) for which the billiard is maximally chaotic and weakly open, respectively. To find the scattering wave function for particles entering and leaving the cavity via the leads, we solve the time-independent Schrödinger equation for ψ under Dirichlet boundary conditions using a plane-wave-expansion method [10], which gives reflection and transmission amplitudes for a given energy. The wave functions are used to compute the different parameters entering the statistics using the explicit expressions stated above.

Figure 1 shows the transmission probability T as a function of wave number k for an incoming wave with transverse mode n in the leads. There is a sequence of overlapping resonances which become broader in the high energy region shown in the right-hand section of figure 1.

For the statistical analysis of the scattering wave functions we select two typical cases: (A) a low energy with only one fully open channel ($n = 1$) for which T reaches unity; (B) a high energy with $n = 4$ and an intermediate value for T. For the statistics the spatial average is taken over the billiard region corresponding to the closed stadium. For convenience this area is set equal to unity.

Figure 2 shows the numerical results for $P(j)$, $P(j_x)$ and $P(j_y)$ together with the analytical predictions in equations (17) and (18). In case (A) there is almost no reflection and hence the system is completely coupled to the open channel. The current statistics shows, however, that $\epsilon = 0.32$; i.e., intermediate between closed and fully open cases. Nevertheless, the numerical results show good agreement with the theory.

Also in the high energy region (B) in figure 1 the probability current distributions are well described by the theory as shown in figure 3. Here $\epsilon = 0.86$ which is close to the exponential Rayleigh (GUE) case $\epsilon = 1$. The transmission is, however, lower than in the previous example. Our numerical results suggest that there is no simple relation between T and ϵ.



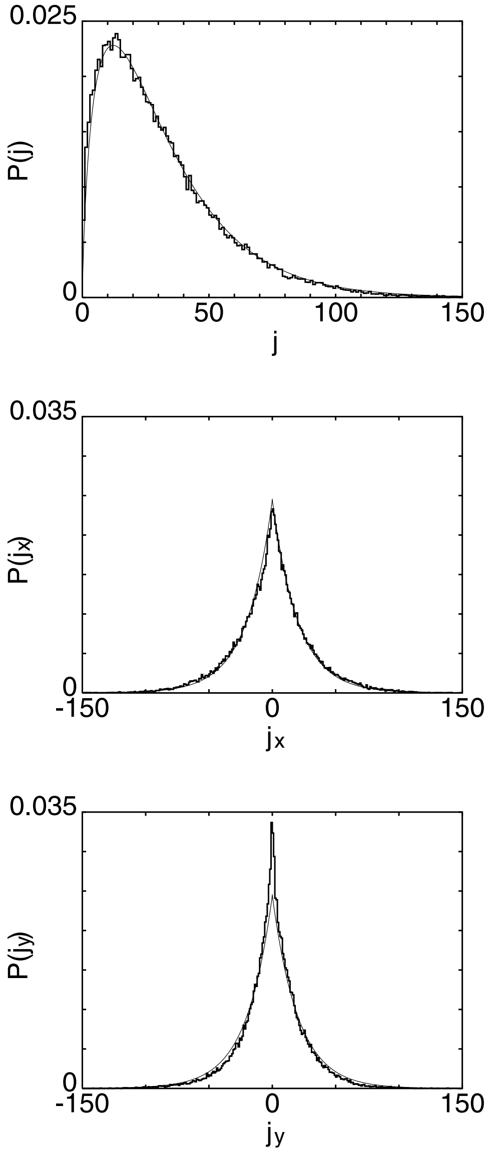


Figure 2. Distribution of probability current density $P(j)$ (top) and its components $P(j_x)$ and $P(j_y)$ (middle and bottom) in the open stadium billiard for case (A) in figure 1. Solid curves show the analytical predictions for $\epsilon = 0.32$. (For convenience $\hbar = 1, m = 1$.)

Figure 3. Same as in figure 2 but for the case (B) for which $\epsilon = 0.86$.

4. Concluding remarks

We have derived the statistical distributions for wave functions, probability current densities and corresponding components for 2D open quantum systems with classically chaotic dynamics. The expressions for the probability currents are universal in the sense that the shape of the distributions is independent of the mixing parameter ϵ; i.e., only the width changes with ϵ. This is in contrast to the wave function statistics that transforms gradually from GOE to GUE type with increasing ϵ. Obviously these ideas carry over into 3D.

The statistics for mesoscopic transport through a chaotic open billiard was also studied numerically with sufficient statistical resolution to compare with the analytical predictions. The results give a numerical verification of the predictions for the probability current density developed here. It also appears that experimental verifications are possible. For example, images of the coherent electron flow through a quantum point contact have been observed in recent experiments [7, 8]. There is also the case of thin microwave resonators [1, 9, 11] where the present theory might be applied to the Poynting vector.

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References

[1] Stöckmann H-J 1999 *Quantum Chaos: An Introduction* (Cambridge: Cambridge University Press)
[2] Zyczkowski K and Lenz G 1991 *Z. Phys.* B **82** 299
[3] Kanzieper E and Freilikher V 1996 *Phys. Rev.* B **54** 8737
[4] Pnini R and Shapiro B 1996 *Phys. Rev.* E **54** R1032
[5] Šeba P, Haake F, Kus M, Barth M, Kuhl U and Stöckmann H-J 1997 *Phys. Rev.* E **56** 2680
[6] Ishio H, Saichev A I, Sadreev A F and Berggren K-F 2001 *Phys. Rev.* E **64** 056208
[7] Topinka M A, LeRoy B J, Shaw S E J, Heller E J, Westervelt R M, Maranowski K D and Gossard A C 2000 *Science* **289** 2323
[8] Topinka M A, LeRoy B J, Westervelt R M, Shaw S E J, Fleischmann R, Heller E J, Maranowski K D and Gossard A C 2001 *Nature* **410** 183
[9] Šeba P, Kuhl U, Barth M and Stöckmann H-J 1999 *J. Phys. A: Math. Gen.* **32** 8225
[10] Nakamura K and Ishio H 1992 *J. Phys. Soc. Jpn.* **61** 3939
[11] Barth M 2001 *PhD Thesis* Philipps-Universität Marburg